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New S function series and non-compact Lie groups

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Abstract. A number of new infinite series of S functions are described in terms of their generating functions and S function content. Applications to the character theory of non-compact Lie groups are noted.

1. Introduction

Schur functions (or S functions for brevity) arose early in the theory of the symmetric group. Generating functions for a number of infinite series of S functions were constructed and shown to be relevant to the characters of regular matrix groups (Littlewood 1950). Further properties of these series were elucidated and proofs improved (McConnell and Newell 1973, Macdonald 1979).

S function series have been used to great effect in obtaining compact expressions for the evaluation of branching rules and Kronecker products for compact Lie groups (King 1975, King *et al* 1981, King and Wybourne 1982, Black *et al* 1983, Black and Wybourne 1983). Similar expressions have been obtained for the discrete series irreps of non-compact Lie groups (Rowe *et al* 1985, King and Wybourne 1985) and to super Lie groups (Dondi and Jarvis 1981, King 1983, Wybourne 1984). In many of these applications infinite series of S functions are symbolically manipulated to yield the desired results. The extensions to non-compact Lie groups (King and Wybourne 1985) revealed the existence of further classes of S function series. In view of these known applications it seems worth recording any new series of S functions.

In this paper we attempt to give a reasonably systematic presentation of infinite S function series and establish a number of new series and their associated properties. We first show how the known series can be obtained either by a simple substitution or as products of simpler series. We note their generating functions and their relationship, in many cases, to the operation of plethysm (Littlewood 1950, Wybourne 1970) or equivalently the wreath product (Read 1968, Thomas 1976, Macdonald 1979). We then develop a new class of hitherto unrecorded series and give their S function content. Finally we indicate how the ideas developed by McConnell and Newell (1973) may be extended to establish the S function content of the new series.

2. The classical S function series

King (1975), following upon the earlier work of Littlewood (1950), chose to designate a set of twelve distinct infinite series of S functions by the capital letters A, B, C, D, E, F, G, H, L, M, P and Q, noting the occurrence of pairs of series that were mutual inverses and others that formed conjugate pairs. The generating functions for these series were given by Littlewood (1950) as was their S function content. These series were multiplicity free in the sense that in any given series every member of the series occurred but once. Four other series, designated R, S, W and V, were established later (King *et al* 1981). We shall refer to these sixteen series as the classical S function series. They are all derivable from the single infinite series of S functions designated as the L series (King 1975).

The L series is associated with the generating function (Littlewood 1950)

$$L=\prod_{i=1}^{\infty}(1-x_i). \tag{1}$$

The terms in L may be expanded as an infinite set of monomials in the x_i, which may in turn be expressed in terms of S functions to give

$$L=\sum_{m=0}^{\infty}(-1)^m\{m\}. \tag{2}$$

The inverse L^{-1} is readily found to involve

$$L^{-1}=\left(\prod_{i=1}^{\infty}(1-x_i)\right)^{-1}=\prod_{m=0}^{\infty}\{m\}. \tag{3}$$

We define the adjoint series $L^{\dagger}$ as the conjugate ($\sim$) inverse or the inverse conjugate of L:

$$L^{\dagger}=(\tilde{L})^{-1}=\tilde{L}^{-1} \tag{4}$$

leading to

$$L^{\dagger}=\prod_{i=1}^{\infty}(1+x_i)=\sum\{1^m\}. \tag{5}$$

Notice that taking the adjoint $\dagger$ is equivalent to the substitution

$$x_i\rightarrow -x_i$$

in $L(x_i)$, which can be viewed as a plethysm:

$$L^{\dagger}=L(-x_i)=(-\{1\})\otimes L. \tag{6}$$

The conjugate of L is also the inverse of $L^{\dagger}$ and hence

$$\tilde{L}=(L^{\dagger})^{-1}=\left(\prod_{i=1}^{\infty}(1+x_i)\right)^{-1}=\sum_m(-1)^m\{m\}. \tag{7}$$

The four properties, identity (I), conjugation ($\sim$), inverse (-1) and adjoint ($\dagger$) form a discrete four-element group with the multiplication table below.

	I	$\sim$	-1	$\dagger$
I	I	$\sim$	-1	$\dagger$
$\sim$	$\sim$	I	$\dagger$	-1
-1	-1	$\dagger$	I	$\sim$
$\dagger$	$\dagger$	-1	$\sim$	I

Having obtained the four L type series we can obtain further series by simple substitution into the L series. Thus under

$$x_i\rightarrow x_ix_j \qquad (i<j) \tag{8}$$

Table 1. Sixteen classical S function series.

Series	King's designation	Generating function	S functions	Relationship to $L(x_i)$	Plethysm
L	L	$\prod_i (1-x_i)$	$\sum_m (-1)^m\{1^m\}$	$L(x_i)$	$\{1\}\otimes L$
L^{-1}	M	$\left(\prod_i (1-x_i)\right)^{-1}$	$\sum_m \{m\}$	$1/L(x_i)$	$\{1\}\otimes L^{-1}$
$\tilde{L}$	P	$\left(\prod_i (1+x_i)\right)^{-1}$	$\sum_m (-1)^m\{m\}$	$1/L(-x_i)$	$(-\{1\})\otimes L$
$L^\dagger$	Q	$\prod_i (1+x_i)$	$\sum_m \{1^m\}$	$L(-x_i)$	$(-\{1\})\otimes L^{-1}$
A	A	$\prod_{i<j} (1-x_ix_j)$	$\sum_\alpha (-1)^{\omega_\alpha/2}\{\alpha\}$	$L(x_ix_j)(i<j)$	$\{1^2\}\otimes L$
A^{-1}	B	$\left(\prod_{i<j} (1-x_ix_j)\right)^{-1}$	$\sum_\beta \{\beta\}$	$1/L(x_ix_j)(i<j)$	$\{1^2\}\otimes L^{-1}$
$\tilde{A}$	C	$\prod_{i\leq j} (1-x_ix_j)$	$\sum_\gamma (-1)^{\omega_\gamma/2}\{\gamma\}$	$L(x_ix_j)(i\leq j)$	$\{2\}\otimes L$
$A^\dagger$	D	$\left(\prod_{i\leq j} (1-x_ix_j)\right)^{-1}$	$\sum_\delta \{\delta\}$	$1/L(x_ix_j)(i\leq j)$	$\{2\}\otimes L^{-1}$
$V=\tilde{V}$	V	$\prod_i (1-x_i^2)$	$\sum_{p,q} (-1)^p\{\overline{p+2q,p}\}$	$L(x_i^2)$	$(\{2\}-\{1^2\})\otimes L$
$V^{-1}=V^\dagger$	W	$\left(\prod_i (1-x_i^2)\right)^{-1}$	$\sum_{p,q} (-1)^p\{p+2q,p\}$	$1/L(x_i^2)$	$(\{2\}-\{1^2\})\otimes L^{-1}$
$E=\tilde{E}$	E	$\prod_i (1-x_i)\prod_{i<j}(1-x_ix_j)$	$\sum_\varepsilon (-1)^{(\omega_\varepsilon+r)/2}\{\varepsilon\}$	LA	
$E^{-1}=E^\dagger$	F	$\left(\prod_i (1-x_i)\prod_{i<j}(1-x_ix_j)\right)^{-1}$	$\sum_\zeta \{\zeta\}$	$L^{-1}A^{-1}$	
$G=\tilde{G}$	G	$\prod_i (1+x_i)\prod_{i<j}(1-x_ix_j)$	$\sum_\varepsilon (-1)^{(\omega_\varepsilon-r)/2}\{\varepsilon\}$	$L^\dagger A$	
$G^{-1}=G^\dagger$	H	$\left(\prod_i (1+x_i)\prod_{i<j}(1-x_ix_j)\right)^{-1}$	$\sum_\zeta (-1)^{\omega_\zeta}\{\zeta\}$	$\tilde{L}A^{-1}$	
$R=\tilde{R}$	R	$\prod (1-x_i)\left(\prod_i (1+x_i)\right)^{-1}$	$\{0\}+2\sum_{a,b}(-1)^{a+b+1}\begin{pmatrix}a\\b\end{pmatrix}$	$L\tilde{L}$	
$R^{-1}=R^\dagger$	S	$\prod_i (1+x_i)\left(\prod_i (1-x_i)\right)^{-1}$	$\{0\}+2\sum_{a,b}\begin{pmatrix}a\\b\end{pmatrix}$	$L^{-1}L^\dagger$	

we have

$$A = \prod_{i<j} (1 - x_i x_j)$$
$$= L(x_i x_j)$$
$$= \{1^2\} \otimes L$$
$$= \sum_{\alpha} (-1)^{\omega_\alpha} \{\alpha\} \tag{9}$$

where in the Frobenius notation (cf Littlewood 1950)

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \cdots a_r \\ a_1+1 & a_2+1 & \cdots a_r+1 \end{pmatrix}. \tag{10}$$

Continuing, we can construct four series A, A^{-1}, $\tilde{A}$ and $A^{\dagger}$.

The substitution

$$x_i \to x_i^2$$

leads to

$$V = \prod_{i=1}^{\infty} (1 - x_i^2)$$
$$= L(x_i^2)$$
$$= (\{2\} - \{1^2\}) \otimes L$$
$$= \sum_{p,q=0}^{\infty} (-1)^p \{p+2q, p\}. \tag{11}$$

In this case we find $V^{\dagger} = V$ and $\tilde{V} = V^{-1}$ and hence the V series is self-adjoint.

Further series E, G and R may be constructed by forming products of the L and A type series to finally yield the sixteen classical S function series given in table 1. The designation of the S function content follows that of King *et al* (1981).

3. Some new S function series

New series can be formed by making more diverse substitutions into existing series. The simplest substitutions arise in changing the sign of the argument in the generating functions of the A and V type to yield the results of table 2. Again such series can be expressed as an S function plethysm (or wreath product). The S function content of the series in table 2 differ essentially by a phase factor from those of the A and V type series given in table 1. For convenience we have associated each of these series with a letter designation based on King's original series assignments. We also note that there are equivalent substitutions that can yield the same series.

Sixteen non-trivial series arise in forming products of L type with A type series. These are grouped in four sets of four in table 3. Members of a given set are related by the operations of the four-element group mentioned earlier. The S function content of these series are non-trivially different from the corresponding series in table 1 and will be discussed in the following section.

Still further series can be found by considering higher-order substitutions as noted in table 4. Again we discuss their S function content in the following section.

Table 2. The A_+ and V_+ type series ($p_2 = \{2\} - \{1^2\}$).

Series	King's designation	Generating function	S functions	Substitution	Plethysm
A_+	A^+	$\prod_{i<j} (1+x_ix_j)$	$\sum_\alpha \{\alpha\}$	$L(-x_ix_j) i<j$	$(-\{1^2\})\otimes L$
				$L^\dagger(x_ix_j) i<j$	$\{1^2\}\otimes L^\dagger$
A_+^{-1}	B^+	$\left(\prod_{i<j} (1+x_ix_j)\right)^{-1}$	$\sum_\beta (-1)^{\omega_\beta}\{\beta\}$	$L^{-1}(-x_ix_j) i<j$	$(-\{1^2\})\otimes L^{-1}$
				$\tilde{L}(x_ix_j) i<j$	$\{1^2\}\otimes L$
$\tilde{A}_+$	C^+	$\prod_{i\leq j} (1+x_ix_j)$	$\sum_\gamma \{\gamma\}$	$L(-x_ix_j) i\leq j$	$(-\{2\})\otimes L$
				$L^\dagger(x_ix_j) i\leq j$	$\{2\}\otimes L$
$A_+^\dagger$	D^+	$\left(\prod_{i\leq j} (1+x_ix_j)\right)^{-1}$	$\sum_\delta (-1)^{\omega_\delta}\{\delta\}$	$L^{-1}(-x_ix_j) i\leq j$	$(-\{2\})L^{-1}$
				$\tilde{L}(x_ix_j) i\leq j$	$\{2\}\otimes \tilde{L}$
$V_+ = V^\dagger$	V^+	$\prod_i (1+x_i^2)$	$\sum_{p,q} (-1)^q\{\widetilde{p+2q, p}\}$	$L(-x_i^2)$	$(-p_2)\otimes L$
				$L^\dagger(x_i^2)$	$p_2\otimes L^\dagger$
$V_+^{-1} = V_+$	W^+	$\left(\prod (1+x_i^2)\right)^{-1}$	$\sum_{p,q} (-1)^q\{p+2q, p\}$	$L^{-1}(-x_i^2)$	$(-p_2)\otimes L^{-1}$
				$\tilde{L}(x_i^2)$	$p_2\otimes \tilde{L}$

4. S function content of sixteen new series

The expansion of a generating function yields an infinite set of monomial symmetric functions that may, in principle, be converted into S functions via the Kostka matrix (Macdonald 1979) to yield the S function content of the series. Littlewood (1950) has sketched the derivation of the S function content of most of the series in table 1. His derivations were improved by McConnell and Newell (1973) and Macdonald (1979). Here we consider the S function content of the sixteen new series listed in table 3, using a variety of approaches based largely on extensions of the methods developed by McConnell and Newell (1973).

The S function content of the series $F^+ = PB^+$ may be found by inspection of the product of the series P with B^+ to yield

$$\tilde{L}A_+^{-1} = PB^+ = F^+ = \sum_\zeta (-1)^{(\omega_\zeta + n_c)/2}\{\zeta\} \tag{12}$$

where n_c is the number of odd columns of the S function $\{\zeta\}$. The content of the conjugate series LD^+ then becomes

$$LA_+^\dagger = LD^+ = \sum_\zeta (-1)^{(\omega_\zeta + n_r)/2}\{\zeta\} \tag{13}$$

where n_r is the number of odd rows of the S function $\{\zeta\}$.

The S function content of the series $E^+ = QA^+$, QC^+ and their conjugates MC^+ and MA^+ are found via the method of McConnell and Newell (1973). In essence the generating function is multiplied by the alternate

$$\Delta(x) = \prod_{i<j} (x_i - x_j) = \det(x_t^{n-s}) \tag{14}$$

Table 3. Sixteen S function series involving products of L type with A_+ type series.

Series	King symbol	Generating function
$L^{\dagger}A_+$	$E^+=QA^+$	$\prod_i(1+x_i)\prod_{i<j}(1+x_ix_j)$
$L^{-1}\tilde{A}_+$	MC^+	$\prod_{i\leq j}(1+x_ix_j)\left(\prod_i(1-x_i)\right)^{-1}$
$\tilde{L}A_+^{-1}$	$F^+=PB^+$	$\left(\prod_i(1+x_i)\prod_{i<j}(1+x_ix_j)\right)^{-1}$
$LA_+^{\dagger}$	LD^+	$\prod_i(1-x_i)\left(\prod_{i\leq j}(1+x_ix_j)\right)^{-1}$
LA_+	LA^+	$\prod_i(1-x_i)\prod_{i<j}(1+x_ix_j)$
$\tilde{L}\tilde{A}_+$	PC^+	$\prod_{i\leq j}(1+x_ix_j)\left(\prod_i(1+x_i)\right)^{-1}$
$L^{-1}A_+^{-1}$	MB^+	$\left(\prod_i(1-x_i)\prod_{i<j}(1+x_ix_j)\right)^{-1}$
$L^{\dagger}A_+^{\dagger}$	QD^+	$\prod_i(1+x_i)\left(\prod_{i\leq j}(1+x_ix_j)\right)^{-1}$
$L^{-1}A_+$	MA^+	$\prod_{i<j}(1+x_ix_j)\left(\prod_i(1-x_i)\right)^{-1}$
$L^{\dagger}\tilde{A}_+$	QC^+	$\prod_i(1+x_i)\prod_{i\leq j}(1+x_ix_j)$
LA_+^{-1}	LB^+	$\prod_i(1-x_i)\left(\prod_{i<j}(1+x_ix_j)\right)^{-1}$
$\tilde{L}A_+$	PD^+	$\left(\prod_i(1+x_i)\prod_{i\leq j}(1+x_ix_j)\right)^{-1}$
$\tilde{L}A_+$	PA^+	$\prod_{i<j}(1+x_ix_j)\left(\prod_i(1+x_i)\right)^{-1}$
$L\tilde{A}_+$	LC^+	$\prod_i(1-x_i)\prod_{i\leq j}(1+x_ix_j)$
$L^{\dagger}A_+^{-1}$	QB^+	$\prod_i(1+x_i)\left(\prod_{i<j}(1+x_ix_j)\right)^{-1}$
$L^{-1}A_+^{\dagger}$	MD^+	$\left(\prod_i(1-x_i)\prod_{i\leq j}(1+x_ix_j)\right)^{-1}$

where s labels the columns and t the rows of an nth order determinant. This yields a Vandermonde determinant that may be split into a sum of determinants. Finally, the S function contents of the series are identified using the S function definition (Littlewood 1950)

$$\{\lambda\}=\sum_{s,t=1}^{n}\det(x_t^{\lambda_s+n-s})/\det(x_t^{n-s}) \tag{15}$$

to give

$$L^{\dagger}A_+=QA^+=E=\sum\left\{\begin{matrix} 0 & 0 & 0 & 0 & \ldots & 0 & \ldots \\ 1 & 1 & 1 & 1 & \ldots & 1 & \ldots \\ & -2 & 4 & -6 & \ldots & (-1)^{s+1}(2s-2) & \ldots \\ & -3 & 5 & -7 & \ldots & (-1)^{s+1}(2s-1) & \ldots \end{matrix}\right\} \tag{16}$$

where the summation is over all non-standard S functions $\{\lambda_1\lambda_2\ldots\lambda_s\ldots\}$ in which $\lambda_1=0$ or 1 only and $\lambda_s(s>1)=0$, 1, $(-1)^{s+1}(2s-2)$ or $(-1)^{s+1}(2s-1)$, where we use

Table 4. Series obtained by a higher-order substitution ($p_3=\{3\}-\{21\}+\{1^3\}$, $p_4=\{4\}+\{31\}+\{2^2\}-(21^2\}-\{1^4\}$).

Generating function	Relationship to $L(x_i)$	Plethysm
$\prod_i (1-x_i^3)$	$L(x_i^3)$	$p_3 \otimes L$
	$L^\dagger(-x_i^3)$	$(-p_3)\otimes L^\dagger$
$\left(\prod_i (1-x_i^3)\right)^{-1}$	$L^{-1}(x_i^3)$	$p_3\otimes L^{-1}$
	$L^\dagger(-x_i^3)$	$(-p_3)\otimes \tilde{L}$
$\left(\prod_i (1+x_i^3)\right)^{-1}$	$\tilde{L}(x_i^3)$	$p_3\otimes \tilde{L}$
	$L^{-1}(-x_i^3)$	$(-p_3)\otimes L^{-1}$
$\prod (1+x_i^3)$	$L^\dagger(x_i^3)$	$p_3\otimes L^\dagger$
	$L(-x_i^3)$	$(-p_3)\otimes L$
$\prod_i (1-x_i^4)$	$L(x_i^4)$	$p_4\otimes L$
	$L^\dagger(-x_i^4)$	$(-p_4)\otimes L^\dagger$
$\left(\prod_i (1-x_i^4)\right)^{-1}$	$L^{-1}(x_i^4)$	$p_4\otimes L^{-1}$
	$\tilde{L}(-x_i^4)$	$(-p_4)\otimes \tilde{L}$
$\prod_i (1+x_i^4)$	$L(-x_i^4)$	$(-p_4)\otimes L$
	$L^\dagger(x_i^4)$	$p_4\otimes L^\dagger$
$\left(\prod (1+x_i^4)\right)^{-1}$	$L^{-1}(-x_i^4)$	$(-p_4)\otimes L^{-1}$
	$\tilde{L}(x_i^4)$	$p_4\otimes \tilde{L}$
$\prod_i (1+x_i+x_i^2)$	$L(-x_i-x_i^2)$	$(-\{1\}-\{2\}+\{1^2\})\otimes L$
	$L^\dagger(x_i+x_i^2)$	$(\{1\}+\{2\}-\{1^2\})\otimes L$
$\left(\prod (1+x_i+x_i^2)\right)^{-1}$	$L^{-1}(-x_i-x_i^2)$	$(-\{1\}-\{2\}+\{1^2\})\otimes L^{-1}$
	$\tilde{L}(x_i+x_i^2)$	$(\{1\}+\{2\}-\{1^2\})\otimes \tilde{L}$
$\prod_i (1-x_i+x_i^2)$	$L(x_i-x_i^2)$	$(\{1\}-\{2\}+\{1^2\})\otimes L$
	$L^\dagger(x_i-x_i^2)$	$(-\{1\}+\{2\}-\{1^2\})\otimes L^\dagger$
$\left(\prod_i (1-x_i+x_i^2)\right)^{-1}$	$\tilde{L}(-x_i+x_i^2)$	$(-\{1\}+\{2\}-\{1^2\})\otimes \tilde{L}$
	$L^{-1}(x_i-x_i^2)$	$(\{1\}-\{2\}+\{1^2\})\otimes L^{-1}$

the convention that the phase factors $(-1)^{s+1}$ are multiplied outside the S function and thus determine the overall sign of the non-standard S function, while

$$L^\dagger\tilde{A}=QC^+=\sum\left\{\begin{matrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ 1 & 1 & 1 & 1 & \dots & 1 & \dots \\ 2 & -4 & 6 & -8 & \dots & (-1)^{s+1}(2s-2) & \dots \\ 3 & -5 & 7 & -9 & \dots & (-1)^{s+1}(2s-1) & \dots \end{matrix}\right\} \tag{17}$$

where the summation is over all non-standard S functions $\{\lambda_1\lambda_2\dots\lambda_s\dots\}$ such that $\lambda_s(s\geq 1)=0, 1, (-1)^{s+1}(2s)$ or $(-1)^{s+1}(2s+1)$.

The non-standard S functions are standardised using the modification rule

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i, \ldots\} = -\{\lambda_1, \lambda_2, \ldots, \lambda_i - 1, \lambda_{i-1} + 1, \ldots\}. \tag{18}$$

The standard S functions may be expressed in Frobenius notation (cf Littlewood 1950),

$$\begin{pmatrix} a_1 & a_2 \ldots a_r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$$

where r is the rank of the S function, leading to

$$L^{\dagger} A_{+} = QA^{+} = E^{+} = \{0\} + \sum \phi(r, \boldsymbol{a}, \boldsymbol{b}) \begin{pmatrix} a_1 & a_2 \ldots a_r \\ b_1 & b_2 \ldots b_r \end{pmatrix} \tag{19}$$

where $\phi(r, \boldsymbol{a}, \boldsymbol{b})$ is a multiplicity factor

$$\begin{aligned} \phi(r, \boldsymbol{a}, \boldsymbol{b}) &= \operatorname{mult}\begin{pmatrix} a_1 & a_2 \ldots a_r \\ b_1 & b_2 \ldots b_r \end{pmatrix} \\ &= \operatorname{mult}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \operatorname{mult}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \ldots \operatorname{mult}\begin{pmatrix} a_i \\ b_i \end{pmatrix} \ldots \operatorname{mult}\begin{pmatrix} a_r \\ b_r \end{pmatrix} \end{aligned} \tag{20}$$

where

$$\operatorname{mult}\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{cases} 0 & \begin{cases} \text{if } a_i > b_i & 1 \leqslant i \leqslant r \\ \text{or } b_i > a_{i-1} & 1 < i \leqslant r \end{cases} \\ 1 & \begin{cases} \text{if } a_i = b_i & 1 \leqslant i \leqslant r \\ \text{or } a_{i-1} = b_i & 1 < i \leqslant r \end{cases} \\ 2 & \begin{cases} \text{if } a_1 < b_1 & \\ \text{or } a_i < b_i < a_{i-1} & 1 < i \leqslant r. \end{cases} \end{cases} \tag{21}$$

Since $\operatorname{mult}\begin{pmatrix} a_i \\ b_i \end{pmatrix} \leqslant 2$ the multiplicity of a term with rank r cannot exceed 2^r, i.e. $\phi(r, \boldsymbol{a}, \boldsymbol{b}) \leqslant 2^r$.

In a similar manner we may establish that

$$L^{\dagger} A_{+} = QC^{+} = \{0\} + \sum \phi(r, \boldsymbol{a}, \boldsymbol{b}) \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} \tag{22}$$

where now

$$\operatorname{mult}\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{cases} 0 & \begin{cases} \text{if } a_i > b_i + 2 \\ \text{or } a_i < b_{i+1} + 2 \end{cases} \\ 1 & \begin{cases} \text{if } b_i + 2 = a_i \\ \text{or } a_i = b_{i+1} + 2 \end{cases} \\ 2 & \begin{cases} \text{if } b_i + 2 > a_i \text{ and } a_i > b_{i+1} + 2 \\ \text{or } a_i > b_{i+1} + 2 \end{cases} \end{cases} \tag{23a}$$

for $i = 1, 2, \ldots, r-1$ and

$$\operatorname{mult}\begin{pmatrix} a_r \\ b_r \end{pmatrix} = \begin{cases} 0 & \text{if } b_{r+2} < a_r \\ 1 & \begin{cases} \text{if } a_r = 0 \text{ or } 1 \\ \text{or } b_r + 2 = a_r \end{cases} \\ 2 & \text{if } b_r + 2 > a_r. \end{cases} \tag{23b}$$

Again we have $\phi(r, \boldsymbol{a}, \boldsymbol{b}) \leqslant 2^r$.

The MC^+ and MA^+ series are conjugates of QA^+ and QC^+ respectively and hence their S function content may be obtained trivially from QA^+ and QC^+ by simply interchanging the a_i and b_i in the Frobenius notation for these two series.

The series LB^+ and its conjugate PD^+ may be evaluated in a similar manner to give

$$LA_+^{-1} = AB = \sum (-1)^{n+\lambda_1+\ldots+\lambda_l} \times \begin{Bmatrix} \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \ldots & \lambda_l & \lambda_l & 0\ldots 0 \\ -(\lambda_1+1) & -(\lambda_1+1) & -(\lambda_2+1) & -(\lambda_2+1) & \ldots & -(\lambda_l+1) & -(\lambda_l+1) & -1\ldots -1 \end{Bmatrix} \tag{24}$$

for $l = 0, 1, 2, \ldots$. The summation is over all λ_i such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_l > 0$ and $n \geqslant 0$ is the number of ones. Conversion of the series to standard form yields a multiplicity free series

$$LA_+^{-1} = LB^+ = \sum (-1)^{m+n}\{\lambda_1^{k_1}\lambda_1^{\varepsilon_1}(\lambda_1-1)^{\varepsilon_1}(\lambda_1-1)^{k_1'}\ldots\lambda_l^{k_l}\lambda_l^{\varepsilon_l}(\lambda_l-1)^{\varepsilon_l}(\lambda_l-1)^{k_l'}, 1^n\} \tag{25}$$

where now $\lambda_1 > \lambda_2 > \ldots > \lambda_l > 1$, $\varepsilon_i = 0$ or 1 with m being the sum of all λ_i for which $\varepsilon_i = 1$ and k_i, $k_i' = 0 \bmod 4$ for $i = 1, \ldots l$.

The series PD^+ is conjugate to LB^+ and its S function content can be expressed as†

$$PD^+ = \sum_\nu (-1)^{(\omega_\nu+n_0)/2}\{\nu\} \tag{26}$$

where if ν_i is even then $\nu_i - \nu_{i+1} = 0, 1 \bmod 4$, while if ν_i is odd $\nu_i - \nu_{2+1} = 1, 2 \bmod 4$ with $i = 1, 2, \ldots$ and n_0 being the number of odd rows in $\{\nu\}$.

The S function content of the series LA^+, PC^+, MB^+ and QD^+ differ from that for $QA^+ = E^+$, MC^+, $PB^+ = F^+$ and LD^+ respectively by a phase $(-1)^{\omega_\lambda}$ where ω_λ is the weight of the partition (λ). Likewise, the S function contents for the series PA^+, LC^+, QB^+ and MD^+ differ from those of MA^+, QC^+, LB^+ and PD^+ again by the phase factor $(-1)^{\omega_\lambda}$. Thus we have established the S function content of all sixteen series given in table 3.

5. *S* function content of other series

We now consider the series listed in table 4. The generating functions of these series involve x_i alone, unlike the sixteen series just considered. The procedure for obtaining their S function content again involves multiplying the generating function by an alternant determinant $\det(x_i^{n-s})$ expanding the result as a sum of determinants and then dividing the result by the alternant. Thus we find

$$\prod_i (1+x_i^k) = \sum \begin{Bmatrix} 0 & 0\ldots 0\ldots 0 \\ k & k\ldots k\ldots k \end{Bmatrix} \tag{27}$$

summed over all non-standard S functions where k is any integer.

If $k = 3$ we have four related series $\Pi(1+x_i^3)$, its conjugate $1/\Pi(1-x_i^3)$, the inverse $1/\Pi(1+x_i^3)$ and the adjoint $\Pi_i(1-x_i^3)$. Converting the non-standard expression

$$\prod_i (1+x_i^3) = \sum \begin{Bmatrix} 0\ldots 0\ldots 0 \\ 3\ldots 3\ldots 3 \end{Bmatrix} \tag{28}$$

into standard S functions, we obtain the content of $\Pi_i(1+x_i^3)$ in which each S function has a weight $3n$, n integer, and every S function involves only the integers 3, 2 and 1.

† The series PD^+ appears in Jozefiak and Weyman (1985).

$\Pi(1-x_i^3)$ differs from the $\Pi_i(1+x_i^3)$ S function content by a phase factor. The series $1/\Pi(1-x_i^3)$ and $1/\Pi(1+x_i^3)$ are conjugate to $\Pi_i(1+x_i^3)$ and $\Pi_i(1-x_i^3)$ respectively. Thus we have

$$\prod_i (1+x_i^3) = \sum_{n,p,q=0}^{\infty} (\{3^n 2^{3p} 1^{3q}\} - \{3^n 2^{3p+1} 1^{3q+1}\}) \tag{29a}$$

$$\begin{aligned}\Big(\prod_i (1-x_i^3)\Big)^{-1} &= \sum_{n,p,q=0}^{\infty} (\{\widetilde{3^n 2^{3p} 1^{3q}}\} - \{\widetilde{3^n 2^{3p+1} 1^{3q+1}}\}) \\ &= \sum_{m,s,t=0}^{\infty} \{m+3s, m+3t, m\} - \{m+3s+2, m+3t+1, m\}\end{aligned} \tag{29b}$$

$$\prod_i (1-x_i^3) = \sum_{n,p,q=0}^{\infty} (-1)^{n+q} (\{3^n 2^{3p} 1^{3q}\} + \{3^n 2^{3p+1} 1^{3q+1}\}) \tag{29c}$$

$$\Big(\prod_i (1+x_i^3)\Big)^{-1} = \sum_{m,s,t=0}^{\infty} (-1)^{m+s+t} (\{m+3s, m+3t, m\} + \{m+3s+2, m+3t+1, m\}). \tag{29d}$$

The corresponding expansions for $k=4$ follow in a similar manner to give the results

$$\begin{aligned}\prod_i (1+x_i^4) = \sum_{m,n,p,q=0}^{\infty} & (-1)^{n+q} (\{4^m 3^{4n} 2^{4p} 1^{4q}\} - \{4^m 3^{4n+1} 2^{2p} 1^{4q+1}\} \\ & + \{4^m 3^{4n+2} 2^{4p+1} 1^{4q}\} + \{4^m 3^{4n} 2^{4p+1} 1^{4q+2}\} + \{4^m 3^{4n+2} 2^{4p+2} 1^{4q+2}\})\end{aligned} \tag{30a}$$

$$\begin{aligned}\Big(\prod_i (1+x_i^4)\Big)^{-1} = \sum_{m,n,p,q=0}^{\infty} & (-1)^{n+q} (\{m+4n+4p+4q, m+4n+4p, m+4n, m\} \\ & - \{m+4n+2p+4q+2, m+4n+2p+1, m+4n+1, m\} \\ & + \{m+4n+4p+4q+3, m+4n+4p+3, m+4n+2, m\} \\ & + \{m+4n+4p+4q+3, m+4n+4p+1, m+4n, m\} \\ & + \{m+4n+4p+4q+6, m+4n+4p+4, m+4n+2, m\})\end{aligned} \tag{30b}$$

$$\begin{aligned}\prod_i (1-x_i^4) = \sum_{m,n,p,q=0}^{\infty} & (-1)^{m} (\{4^m 3^{4n} 2^{4p} 1^{4q}\} + (-1)^q \{4^m 3^{4n+1} 2^{2p} 1^{4q+1}\} \\ & + \{4^m 3^{4n+2} 2^{4p+1} 1^{4q}\} - \{4^m 3^{4n} 2^{4p+1} 1^{4q+2}\} - \{4^m 3^{4p+2} 2^{4p+2} 1^{4q+2}\})\end{aligned} \tag{30c}$$

$$\begin{aligned}\Big(\prod_i (1-x_i^4)\Big)^{-1} = \sum_{m,n,p,q=0} & (-1)^{m} (\{m+4n+4p+4q, m+4n+4p, m+4n, m\} \\ & + (-1)^q \{m+4n+2p+4q+2, m+4n+2p+1, m+4n+1, m\} \\ & + \{m+4n+4p+4q+3, m+4n+4p+3, m+4n+2, m\} \\ & - \{m+4n+4p+4q+3, m+4n+4p+1, m+4n, m\} \\ & - \{m+4n+4p+4q+6, m+4n+4p+4, m+4n+2, m\}).\end{aligned}$$

Applying the same techniques we also obtain

$$\prod_i (1+x_i+x_i^2) = \sum \begin{Bmatrix} 0 & 0\ldots 0 \ldots 0 \\ 1 & 1 \ldots 1 \ldots 1 \\ 2 & 2 \ldots 2 \ldots 2 \end{Bmatrix} \tag{31}$$

summed over all non-standard S functions $\{\lambda\}$ with $\lambda_i=0,1$ or 2 for $i=1,2,\ldots$. Similarly

$$\prod_i(1-x_i+x_i^2)=\sum\left\{\begin{matrix}0 & 0\ldots\ 0\ldots\ 0\\ -1 & -1\ldots-1\ldots-1\\ 2 & 2\ldots\ 2\ldots\ 2\end{matrix}\right\}. \tag{32}$$

Converting these results into standard form yields

$$\prod_i(1+x_i+x_i^2)=\sum_{p,q=0}^{\infty}\phi(q)\{2^p1^q\} \tag{33a}$$

where the phase factor $\phi(q)$ is given by

$$\phi(q)=\begin{cases}+1 & \text{if } q=0,1 \bmod 6\\ \ \ 0 & \text{if } q=2,5 \bmod 6.\\ -1 & \text{if } q=3,4 \bmod 6\end{cases} \tag{33b}$$

$\Pi_i(1-x_i+x_i^2)$ differs from $\Pi_i(1+x_i+x_i^2)$ by a phase factor $(-1)^q$ to give

$$\prod(1-x_i+x_i^2)=\sum_{p,q=0}^{\infty}\theta(q)\{2^p1^q\} \tag{34a}$$

where

$$\theta(q)=\begin{cases}+1 & \text{if } q=0 \bmod 3\\ \ \ 0 & \text{if } q=2 \bmod 3\\ -1 & \text{if } q=1 \bmod 3.\end{cases} \tag{34b}$$

The series from $1/\Pi(1-x_i+x_i^2)$ and $1/\Pi_i(1+x_i+x_i^2)$ are conjugate to $\Pi_i(1+x_i+x_i^2)$ and $\Pi_i(1-x_i+x_i^2)$ respectively and hence

$$\left(\prod_i(1-x_i+x_i^2)\right)^{-1}=\sum_{p,q=0}^{\infty}\phi(q)\{p+q,p\} \tag{35a}$$

and

$$\left(\prod_i(1+x_i+x_i^2)\right)^{-1}=\sum_{p,q=0}^{\infty}\theta(q)\{p+q,p\}. \tag{35b}$$

Thus the S function content of all the series in table 4 are found.

6. *S* function series identities

S function series identities play an important role in the symbolic manipulation of S function series in deriving formulae for Kronecker products, branching rules, etc. Many identities follow from simple inspection of the relevant generating functions. Many of these identities have already appeared in this paper and will not be repeated. A number of additional identities arise leading to summations over S functions such as

$$A\tilde{A}=A^+C^+=\sum_\zeta\{\zeta\}\{\zeta\} \tag{36a}$$

$$A^{-1}A^{\dagger}=B^+D^+=\sum_\zeta(-1)^{\omega_\zeta}\{\zeta\}\{\zeta\} \tag{36b}$$

leading to

$$A\tilde{A}\cdot\{\zeta\}=\sum_{\zeta}(-1)^{\omega_\zeta}\{\zeta\}\cdot\{\zeta/\xi\} \tag{37a}$$

$$A^{-1}A^{\dagger}\cdot\{\xi\}=\sum_{\zeta}\{\zeta\}\cdot\{\tilde{\zeta}/\xi\}. \tag{37b}$$

We note that the identity for $A\tilde{A}$ arises in the character expression for the non-compact Lie group $U(p, q)$ (King and Wybourne 1985).

7. Application to Sp(2*N*, *R*) symmetrised products

The infinite-dimensional holomorphic discrete series of unirreps of the non-compact group $Sp(2N, R)$ may be labelled as $\langle\{\mu\}\rangle$ where (μ) is a partition into not more than N parts (Rowe *et al* 1985, King and Wybourne 1985).

Under the restriction $Sp(2N, R)\downarrow U(N)$ we have for such unirreps

$$\langle\{\mu\}\rangle\downarrow\{\mu\}\cdot D \tag{38}$$

where the S function product is evaluated in $U(N)$. Inversely

$$\{\mu\}\uparrow\langle\{\mu\}\cdot C\rangle \tag{39}$$

leading to the Kronecker product evaluation

$$\langle\{\mu\}\rangle\alpha\langle\{\nu\}\rangle\downarrow\{\mu\}\cdot\{\nu\}\cdot DD\uparrow\{\mu\}\cdot\{\nu\}DDC \tag{40}$$

and hence

$$\langle\{\mu\}\rangle\alpha\langle\{\nu\}\rangle=\langle\{\mu\}\rangle\cdot\{\nu\}\cdot D\rangle \tag{41}$$

where again the S function products are evaluated in $U(N)$.

An expression for the symmetrised powers of a holomorphic unirrep of $Sp(2N, R)$ may be obtained by first noting that

$$\langle\{0\}\rangle\otimes\{\nu\}=\langle(D\otimes\{\nu\})\cdot C\rangle=g^{\rho}_{\nu}\langle\{\rho\}\rangle \tag{42}$$

leading to

$$\langle\{\lambda\}\rangle\otimes\{\nu\}=\sum_{\nu,\rho}g^{\rho}_{\nu}\langle(\{\lambda\}\otimes\{\mu\}\circ\nu)\cdot\{\rho\}\rangle \tag{43}$$

with all S function products and plethysms occurring in $U(N)$. The key to the use of (42) is the evaluation of the terms $\{\rho\}$ in

$$D\otimes\{\nu\}\cdot C=g^{\rho}_{\nu}\{\rho\}. \tag{44}$$

For the particular case of the Kronecker square we have to evaluate

$$D\otimes\{2\}\cdot C \qquad D\otimes\{1^2\}\cdot C. \tag{45}$$

Noting that

$$\{2\}=\tfrac{1}{2}(p_2+p_{12}) \qquad \{1^2\}=\tfrac{1}{2}(p_{12}-p_2) \tag{46}$$

and that $DC=1$ we readily find

$$D\otimes\{2\}\cdot C=\tfrac{1}{2}(D+D^{+})=D_{+}$$

$$D\otimes\{1^2\}\cdot C=\tfrac{1}{2}(D-D^{+})=D_{-} \tag{47}$$

where

$$D_\pm = \sum |\delta_\pm| \tag{48}$$

with δ_+ being any partition into even parts only of the integers $4p$ and δ_- of the integers $4p+2$. Thus

$$\langle\{\lambda\}\rangle \otimes \{2\} = \langle\{\lambda\}\otimes\{2\}\cdot D_+\rangle + \langle\{\lambda\}\otimes\{1^2\}\cdot D_-\rangle \tag{49a}$$

and

$$\langle\{\lambda\}\rangle \otimes \{1^2\} = \langle\{\lambda\}\otimes\{1^2\}\cdot D_+\rangle + \langle\{\lambda\}\otimes\{2\}\cdot D_-\rangle. \tag{49b}$$

8. Concluding remarks

We have obtained the S function content of a number of additional S function series. The application of these series to problems in the character theory of groups remains to be explored in detail though the relevance of some of the series to non-compact groups is already apparent.

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